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INVESTIGATING POSITIVE PERIODIC SOLUTIONS IN SUPERLINEAR FIRST-ORDER DIFFERENTIAL SYSTEMS THROUGH GLOBAL BIFURCATION THEORY

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Abstract: This paper investigates the superlinear first-order differential systems of the form u′(t)+a(t)u(t)=λb(t)f(v(t−τ(t))), t∈R, v′(t)+a(t)v(t)=λb(t)g(u(t−τ(t))), t∈R, where λ is a parameter and a, b, τ, f, and g satisfy certain assumptions. The aim of this study is to explore the existence of positive periodic solutions for this system and use the bifurcation theory to demonstrate the presence of an unbounded component of positive solutions. The authors use a new technique to directly prove that the component must bifurcate from infinity at λ = 0 without additional conditions of f, g. The research has significant applications in areas such as physics, information, chemistry, engineering, economics, and mathematical biology. The authors also discuss the asymptotic problem in an abstract setting and apply the result to the proof of the main theorem. Overall, the paper contributes to the study of differential equations under superlinear or sublinear conditions and provides useful information for the numerical solutions of such equations

Keywords: superlinear differential system, bifurcation theory, positive periodic solutions, unbounded component, asymptotic problem, numerical solutions

Introduction

We are concerned with $\begin{cases} u'(t)+a(t)u(t)=\lambda b(t)f(v(t-\tau(t))), & t\in\mathbb{R} \\ v'(t)+a(t)v(t)=\lambda b(t)g(u(t-\tau(t))), & t\in\mathbb{R}, \end{cases}$ the first-order system of the form

(1.1) where $\lambda \in$ R is a parameter, a,b,τ and f,g satisfy the assumptions:

(H1) $\tau \in C$(R,R) is an ω-periodic function, $a,b \in C$(R,[0,+∞)) are two ω-periodic functions such that $\int_0^\omega a(t)\mathrm{d}t>0,\ \int_0^\omega b(t)\mathrm{d}t>0$;

(H2) $f,g \in C$(R,(0,+∞)) are nondecreasing functions;

(H3) $\lim\limits_{s\to+\infty}\frac{f(s)}{s}=+\infty=\lim\limits_{s\to+\infty}\frac{g(s)}{s}$.

The corresponding scalar equation of the form (1.1) attracts much attention from mathematicians. Many authors devote themselves to exploring the existence of periodic solutions of this kind of equation and some

excellent results have been achieved, see [4, 5, 7–12, 15, 17]. For instance, Cheng and Zhang [4] considered the existence of positive $\omega-$periodic solution for equation

$u'(t) + a(t)u(t) = \lambda b(t)f(u(t-\tau(t)))$, $t \in \mathbb{R}$, (1.2) where λ is a parameter. By using the fixed point theorem in cones, they obtained the following result. **Theorem 1.1** *([4, Thm 2.3]) Assume a,b,τ satisfy (H1), f satisfies (H2) and (H3), then for any* $\lambda \in (0,\bar{\lambda})$*, equation (1.2) has at least two positive ω−periodic solutions, where*

$$\bar{\lambda} = \frac{1}{B}\sup_{m>0}\frac{m}{\max\limits_{0\leqslant x\leqslant m} f(x)},$$

$$B = \max_{0\leqslant t\leqslant \omega}\int_0^{\omega} G(t,s)b(s)ds,$$ *and G(t,s) is expressed as (1.6) below.*

Problem (1.1) represents a class of differential systems with delay. The delay differential equation is mainly used to describe the current and past history of state power systems, so it is in the physical, information, chemistry, engineering, economics, and mathematical biology and other fields that have important applications. Up to now, many scholars have made good research results in this aspect, see [16, 20], other similar research can be found in [1, 2, 14].

However, there are only a few papers concerning the existence of positive periodic solutions for first-order systems, we can refer to [3, 13, 18] which, via the fixed point theorem in the cones and the Schauder's fixed point. It is worth mentioning that due to the limitations of the tools they used, all the results mentioned above did not provide information on the global behavior of positive periodic solution sets. However, this global behavior is very useful for computing the numerical solutions of differential equations as it can be used to guide numerical work. For example, it can be used to estimate the *u*-interval in advance in applying the finite difference method and when applying the shooting method, it can be used to restrict the range of initial values that need to be considered.

In [6], Chhetri and Girg studied a system of the semilinear equation of the form

$$\begin{cases} -\Delta u = \lambda f(v), \text{ in } \Omega, \\ -\Delta v = \lambda g(u), \text{ in } \Omega, \\ u = 0 = v, \text{ on } \partial\Omega, \end{cases} \quad (1.3)$$

where $\lambda \in \mathbb{R}$ is a parameter and $\Omega \subset \mathbb{R}^N, N \geqslant 2$ is a bounded domain with $C^{2,\eta}$-boundary for some $\eta \in (0,1)$, the nonlinearities f and g satisfy (H2) and (H3). Under these assumptions, they obtained the global behavior of positive solution sets of (1.3) by using the global bifurcation theorem. To be more precise, they obtained a component of positive solutions for (1.3), emanating from the origin, which is bounded in positive λ-direction. If in addition, Ω is convex, and $f,g \in C^1$ satisfy the certain subcritical condition, they showed that the component must bifurcate from infinity at $\lambda = 0$.

Motivated by [4] and [6], we attempt to study the global bifurcation behavior of positive ω−perodic solutions for problem (1.1) and give a similar result to [6]. Compared to [6], the innovation of our result is that we use a new technique, which can directly prove that the component must bifurcate from infinity at $\lambda = 0$ without additional conditions of f,g in [6]. This technique plays an important role in applying bifurcation theory to study the global structure of solution sets of differential equations under superlinear or sublinear conditions because it can directly obtain the limits of a sequence of connected sets, which are the connected branches of the solution set of the problem under study.

let $X_0 = \{u \in C(\mathbb{R},\mathbb{R}) \mid u(t) = u(t+\omega)\}$, $E_0 = C^1(\mathbb{R},\mathbb{R})$, denote $X = X_0 \times X_0$ and $E = E_0 \times E_0$, it is easy to know X and E are Banach space endowed with the norm $\|(u_1,u_2)\|_X = \|u_1\|_C + \|u_2\|_C$ and $\|(u_1,u_2)\| = \|u_1\|_{C^1} + \|u_2\|_{C^1}$ respectively, where $\|u\|_C = \max\limits_{0\leqslant t\leqslant\omega} u(t),\ \|u\|_{C^1} = \|u\|_C + \|u'\|_C$.

We denote Π of the form

$\Pi = \{(\lambda,(u,v)) \in R \times E \mid (\lambda,(u,v))$ solution of (1.1)$\}$.

If $(\lambda,(u,v)) \in \Pi$ and $u > 0$, $v > 0$, then we say that $(\lambda,(u,v))$ is a positive solution of (1.1). By a continuum of solutions of (1.1) we mean a subset $K \subset \Pi$ which is closed and connected. By a component of solutions set Π we mean a continuum which is maximal with respect to inclusion ordering. We say that λ is a ∞ bifurcation point from infinity if the solution set Π contains a sequence $(\lambda_n,(u_n,v_n))$ such that $\lambda_n \to \lambda_\infty$ and $\|(u_n,v_n)\| \to +\infty$ as $n \to +\infty$. We say that a continuum C bifurcates from infinity at $\lambda \in R$ if there exists a sequence of solutions $(\lambda_n,(u_n,v_n)) \in C$ such that $\lambda_n \to \lambda_\infty$ and $\|(u_n,v_n)\| \to +\infty$ as $n \to +\infty$.

It is easy to see that (1.1) can be written as

$$u(t) = \lambda \int_t^{t+\omega} G(t,s)b(t)f(v(s-\tau(s)))ds, \quad (1.4)$$

$$v(t) = \lambda \int_t^{t+\omega} G(t,s)b(t)g(u(s-\tau(s)))ds, \quad (1.5)$$

where

$$G(t,s) = \frac{e^{\int_t^s a(\theta)d\theta}}{e^{\int_0^\omega a(\theta)d\theta} - 1}, \qquad s \in [t,t+\omega]. \quad (1.6)$$

Notice that $\int_0^\omega a(\theta)d\theta > 0$, we have $\frac{1}{\sigma-1} \leqslant G(t,s) \leqslant \frac{\sigma}{\sigma-1}$,

where $\sigma = e^{\int_0^\omega a(\theta)d\theta}$ and $0 < 1/\sigma < 1.$

Define that K is a cone in X_0 by

$$K := \{u \in X_0 \mid u(t) \geqslant 0,\ u(t) \geqslant \frac{1}{\sigma}\|u\|,\ t \in \mathbb{R}\}.$$

Obviously, K is a total cone.

For $u \in K$, consider the corresponding linear eigenvalue problem of (1.1) as follows:

$$u'(t) + a(t)u(t) = \lambda b(t)u(t), \qquad t \in R. \quad (1.7)$$

Denote the operator $L : E_0 \cap K \to C(R,R)$ by

$$Lw := w' + a(t)w,\ w \in E_0 \cap K.$$

A direct result of Krein-Rutman theorem, (1.7) has a unique eigenvalue μ_1, which is positive and simple, and the corresponding eigenfunction φ_1 is positive. Moreover, μ_1 is also an eigenvalue of L_*, that is, there exists a positive φ_1^* such that

$$L^*\varphi_1^* = \mu_1\varphi_1^*, \quad (1.8)$$

where L_* is the conjugate operator of L.

We first state a nonexistence result, which holds under weaker assumptions than (H1)-(H3).

Theorem 1.2 *Suppose there exist two constants $\alpha,\beta > 0$ such that $f(s),g(s) > \alpha s + \beta$ for all $s \in R$. Then there are no solutions of (1.1) for $\lambda \geqslant \lambda_* \overset{def}{=} \mu_1/\alpha$.*

The following theorem is the existence result.

Theorem 1.3 *Let (H1)-(H3) hold. Then there exists an unbounded component $C \subset \Pi$ satisfying the following:*

(a) *$(\lambda,(u,v)) \in C$ is positive whenever $\lambda \in (0,\lambda_*)$;*

(b) *(0,(0,0)) is the only element belonging to C with $\lambda = 0$;*

(c) *$Proj_{\lambda\in[0,+\infty)}C \overset{def}{=} \{\lambda \in [0,+\infty) \mid \exists (u,v) \in E$ with $(\lambda,(u,v)) \in C\} \subset [0,\lambda^*)$;*

(d) *any sequence* $(\lambda_n,(u_n,v_n)) \in C$ *such that* $\|(u_n,v_n)\| \to +\infty$ *as* $n \to +\infty$ *and* $\lambda_n > 0$ *must satisfy* $\lambda_n \to 0^+$ *as* $n \to +\infty$.

Remark 1.4 *We may conclude the number of positive solutions of (1.1) from Theorem 1.3:*

(i) *(1.1) has no positive solution for* $\lambda \geqslant \lambda^*$;

(ii) *there exists* $\lambda_* < \lambda^*$ *such that (1.1) has at least two positive solutions for each* $\lambda \in (0,\lambda_*)$.

Example 1.5 *Consider the first-order differential system of the form:*

$$\begin{cases} u'(t) + (\sin t + 1)u(t) = \lambda(\cos t + 1)e^{v(\gamma(t))}, & t \in \mathbb{R} \\ v'(t) + (\sin t + 1)v(t) = \lambda(\cos t + 1)e^{(u(\gamma(t)))^2} & t \in \mathbb{R}, \end{cases},$$

Obviously, $a(t) = \sin t + 1$, $b(t) = \cos t + 1$, $\tau(t) = \sin t$, $f(s) = e^s$, $g(s) = e^{s^2}$, $\gamma(t) = t - \sin t$, *and satisfy the hypotheses of (H1)-(H3).*

1. Preliminary results

To prove Theorem 1.3, first, we approximate the superlinear nonlinearities f and g by a sequence of asymptotically positively homogeneous nonlinearities f_n and g_n, respectively. Then we will discuss this asymptotic problem in an abstract setting and apply the result to the proof of Theorem 1.3. In order to discuss this auxiliary, we consider the first eigenpair of the following eigenvalue problem

$$\begin{cases} w_1'(t) + a(t)w_1(t) = \lambda\theta_1 b(t)w_2(t), & t \in \mathbb{R} \\ w_2'(t) + a(t)w_2(t) = \lambda\theta_2 b(t)w_1(t), & t \in \mathbb{R}, \end{cases}$$

,

(2.1) where $w_1,w_2 \in X_0 \cap E_0$, θ_1,θ_2 are positive constants.

For convenience, the eigenvalue problem (2.1) can be read as follows:

$$L\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \lambda b(t)\begin{bmatrix} 0 & \theta_1 \\ \theta_2 & 0 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Note that $-\sqrt{\theta_1\theta_2}$ and $\sqrt{\theta_1\theta_2}$,

$$\begin{bmatrix} -\sqrt{\theta_1} \\ \sqrt{\theta_2} \end{bmatrix} \text{ and } \begin{bmatrix} \sqrt{\theta_1} \\ \sqrt{\theta_2} \end{bmatrix}$$

are respective eigenvalues and eigenvectors of the coefficient matrix. Taking into account the linearity of operator L, we infer that

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -\sqrt{\theta_1} & \sqrt{\theta_1} \\ \sqrt{\theta_2} & \sqrt{\theta_2} \end{bmatrix}^{-1}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

satisfies

$$L\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \lambda b(t)\begin{bmatrix} -\sqrt{\theta_1\theta_2} & 0 \\ 0 & \sqrt{\theta_1\theta_2} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

The equations of this system are not coupled and it is obvious that $z_1 = 0$6if $-\lambda\sqrt{\theta_1\theta_2} = \mu_1$ and $z1 = \varphi_1$. On the other hand, $z_2 = 0$6 if $\lambda\sqrt{\theta_1\theta_2} = \mu_1$ and $z_1 = \varphi_1$. Therefore, $z_1 = 0$6implies $z_2 = 0$ and $z_2 = 0$6 implies $z_1 = 0$. Hence, the eigenfunctions of corresponding to $\lambda = -\frac{\mu_1}{\sqrt{\theta_1\theta_2}}$ are

$$\begin{bmatrix} -\sqrt{\theta_1} & \sqrt{\theta_1} \\ \sqrt{\theta_2} & \sqrt{\theta_2} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{\theta_1} \\ \sqrt{\theta_2} \end{bmatrix} \varphi_1,$$

which is $(-\sqrt{\theta_1}\varphi_1,\sqrt{\theta_2}\varphi_1)$. Analogously, we get the eigenfunctions corresponding to $\lambda = \frac{\mu_1}{\sqrt{\theta_1\theta_2}}$ as $(\sqrt{\theta_1}\varphi_1,\sqrt{\theta_2}\varphi_1)$. Note that φ_1 is an eigenfunction of (1.7) which is positive. Thus $\nu_1 \overset{\text{def}}{=} \frac{\mu_1}{\sqrt{\theta_1\theta_2}}$ is a unique simple eigenvalue of (2.1) such that both components of its eigenfunctions, $(\sqrt{\theta_1}\varphi_1,\sqrt{\theta_2}\varphi_2) \in E$, are positive in R.

Now, consider an asymptotically positively homogeneous system of the form

$$\begin{cases} u'(t) + a(t)u(t) = \lambda\theta_1 b(t)v^+(t) + \lambda b(t)\tilde{f}(v(t)), & t \in \mathbb{R}, \\ v'(t) + a(t)v(t) = \lambda\theta_2 b(t)u^+(t) + \lambda b(t)\tilde{g}(u(t)), & t \in \mathbb{R}, \end{cases} \quad (2.2)$$

where $u,v \in X_0 \cap E_0$, $x^+ \overset{\text{def}}{=} \max\{0,x\}$ is the positive part of x, θ_1,θ_2 are in (2.1). The nonlinear perturbations $\tilde{f},\tilde{g} : \mathbb{R} \to \mathbb{R}$ satisfy the following assumptions: ∈

(A1) $\tilde{f}$ and $\tilde{g}$ are continuous and bounded functions;

(A2) $\theta_1 x^+ + \tilde{f}(x), \theta_2 x^+ + \tilde{g}(x) > 0$ for all x R.

Let

$T = \{(\lambda,(u,v)) \in \mathbb{R} \times E \mid (\lambda,(u,v))$ solution of (2.2)$\}$, then

we prove the following bifurcation result.

Lemma 2.1 *Let (A1)–(A2) hold. Then ν_1 is the unique bifurcation point from infinity for (2.2). Moreover, there exists a continuum $D \subset T$ bifurcating from infinity at ν_1 and satisfies the following:*

(i) *if $(\lambda,(u,v)) \in D$ and $\lambda > 0$ then $u > 0$ and $v > 0$;*

(ii) *for $\lambda = 0, (u,v) = (0,0)$ is the only solution of (2.2) and $(0,(0,0)) \in D$; def*

(iii) *$Proj_{\lambda}C = \{\lambda \in \mathbb{R} \mid \exists (u,v) \in E$ with $(\lambda,(u,v)) \in D\}$ is bounded from above and unbounded from below.*

Proposition 2.2 *([21, Thm. 14.D]) Let Y be a Banach space with $Y \neq \{0\}$ and let $F : Y \to Y$ be compact. Then the solution component $C \subset \mathbb{R} \times Y$ of the equation $x = \lambda F(x)$ which contains $(0,0) \in \mathbb{R} \times Y$ is unbounded as are both subsets def*

$C_{\pm} = C \cap (\mathbb{R}_{\pm} \times Y)$, *where* $\mathbb{R}_+ \overset{def}{=} [0,\infty)$

and $\mathbb{R}_- \overset{def}{=} (-\infty,0]$.

Definition 2.3 *([19]) Let Z be a Banach space and $\{C_n \mid n = 1,2,\cdots\}$ be a certain infinite collection of subset of Z. Then the superior limit of D of $\{C_n\}$ is defined by*

$$\mathcal{D} := \limsup_{n\to\infty} C_n = \{x \in Z \mid \exists \{n_i\} \subset \mathbb{N} \text{ and } x_{n_i} \in C_{n_i}, \text{ such that } x_{n_i} \to x\}.$$

Lemma 2.4 *([19])Let Z be a Banach space with the norm $\|\cdot\|_Z$, let $\{C_n\}$ be a family of closed subsets of Z. Assume that:*

(i) *there exist $z_n \in C_n$, $n = 1,2,\cdots$, and $z_* \in Z$, such that $z_n \to z_*$;*

(ii) *$d_n = \sup\{\|x\|_Z \mid x \in C_n\} = \infty$;*

(iii) *for every $R > 0$, $(\bigcup_{n=1}^{\infty} C_n) \cap B_R$ is a relatively compact set of Z, where $B_R = \{x \in Z \mid \|x\|_Z \leqslant R\}$, then there exists an unbounded component C in D and $z_* \in C$.*

2. Proof of main results

We define an inner product on $E_0 \cap X_0$ by

$$\langle x,y \rangle := \int_0^1 x(t)y(t)dt, \quad x,y \in E_0 \cap X_0.$$

(3.1)

0

Proof of Theorem 1.2 According to (1.4)-(1.5) and the positivity of f and g, all solutions $(\lambda,(u,v))$ of (1.5) with $\lambda > 0$ must satisfy $u,v > 0$. Let $(\lambda,(u,v))$ be a solution of (1.1) with $\lambda > 0$, then due to the assumptions of f and g, we get

$$(u+v)' + a(t)(u+v) > \lambda b(t)(\alpha(u+v)+2\beta),\ t \in \mathrm{R}.$$

Denoting $z \overset{\text{def}}{=} u+v$, we see that $z > 0$ and $z' + a(t)z > \lambda b(t)(\alpha z + 2\beta)$, then by combining (3.1) and (1.8), we get

$$\int_0^1 (z' + a(t)z)\varphi_1^* \mathrm{d}t = \langle Lz, \varphi_1^* \rangle = \langle z, L^*\varphi_1^* \rangle = \mu_1 \langle z, b(t)\varphi_1^* \rangle,$$

and

$$\int_0^1 (z' + a(t)z)\varphi_1^* \mathrm{d}t > \lambda \int_0^1 b(t)(\alpha z + 2\beta)\varphi_1^* \mathrm{d}t > \lambda\alpha \int_0^1 b(t) z \varphi_1^* \mathrm{d}t = \lambda\alpha \langle z, b(t)\varphi_1^* \rangle,$$

then

$$\mu_1 \langle z, b(t)\varphi_1^* \rangle > \lambda\alpha \langle z, b(t)\varphi_1^* \rangle.$$

This combines with (H1) and the positivity of z, φ_1^*, we must have $\lambda < \mu_1/\alpha$. Therefore, (1.1) has no solution for $\lambda \geqslant \lambda^* \overset{\text{def}}{=} \mu_1/\alpha$. 2

Proof of Lemma 2.1 The operator equation corresponding to the system (2.2) is

$(u,v) = \lambda L^+(u,v) + \lambda H(u,v)$, (3.2) where $L^+ : E \to E$ denotes the mapping $(u,v) \mapsto L^{-1}(\theta_1 b(t)v^+, \theta_2 b(t)u^+)$ and $H : E \to E$ denotes the mapping $(u,v) \mapsto L^{-1}(b(t)\tilde{f}(v), b(t)\tilde{g}(u))$.

Then L^+ is not linear but both L^+ and H are continuous and compact. Moreover, since f,g and b are bounded, H satisfies ee

$$\lim_{\|(u,v)\| \to +\infty} \frac{\|H(u,v)\|}{\|(u,v)\|} = 0,$$

(3.3) which is crucial in establishing a version of a global bifurcation result for (3.2).

The following proposition shows that the unique possible bifurcation point from infinity for (3.2) is ν_1.

Proposition 3.1 *If ν_∞ is a bifurcation point from infinity for (3.2), then $\nu_\infty = \nu_1$. Moreover, for any sequence $(\lambda_j,(u_j,v_j)) \in \mathrm{R} \times E$ with $\lambda_j \to \nu_1$ and $\|(u_j,v_j)\| \to +\infty$ as $j \to +\infty$, there exists a subsequence $(\lambda_{jk},(u_{jk},v_{jk}))$ such that*

$$\lim_{j_k \to +\infty} \frac{(u_{j_k}, v_{j_k})}{\|(u_{j_k}, v_{j_k})\|} = \frac{(\sqrt{\theta_1}\varphi_1, \sqrt{\theta_2}\varphi_1)}{\|(\sqrt{\theta_1}\varphi_1, \sqrt{\theta_2}\varphi_1)\|},$$

where the convergence is in E.

Proof Let $(\lambda_j,(u_j,v_j)) \in \mathrm{R} \times E$ be a solution of (2.2) such that $\|(u_j,v_j)\| \to +\infty$ and $\lambda_j \to \nu_\infty$. Then $(\widehat{u}_j, \widehat{v}_j) = \frac{(u_j,v_j)}{\|(u_j,v_j)\|}$ satisfies

$$\widehat{u}_j = \lambda_j L^{-1}\left(\theta_1 b(t)\widehat{v}_j^+ + b(t)\frac{\widetilde{f}(v_j)}{\|(u_j,v_j)\|}\right),$$

$$\widehat{v}_j = \lambda_j L^{-1}\left(\theta_2 b(t)\widehat{u}_j^+ + b(t)\frac{\widetilde{g}(u_j)}{\|(u_j,v_j)\|}\right),$$

or equivalently satisfies

$$(\widehat{u}_j, \widehat{v}_j) = \lambda_j L^+(\widehat{u}_j, \widehat{v}_j) + \frac{\lambda_j H(u_j,v_j)}{\|(u_j,v_j)\|}.$$

It then follows from (3.3) that the right hand side is bounded in X (independent of j). Hence ku_jkC1 and kvb0$_j$kC1 are bounded (independent ofbb j), and there exists subsequence of ub$_j$ and vb$_j$ converging to $u$bb and vb in
E . Therefore $(\nu\infty,(u,v)) \in \mathbb{R} \times (E \cap X)$ satisfies
$$\begin{cases} \widehat{u}' + a(t)\widehat{u} = \nu_\infty \theta_1 b(t)\widehat{v}^+, & t \in \mathbb{R}, \\ \widehat{v}' + a(t)\widehat{v} = \nu_\infty \theta_2 b(t)\widehat{u}^+, & t \in \mathbb{R}. \end{cases}$$
Suppose $\nu\infty \leqslant 0$. Since $v^+ \geqslant 0$, it follows (1.4) that $u \equiv 0$ and repeating the same argument we get $v \equiv 0$ as well. This leads to a contradiction sincebk(u,bb$v$b+)k = 1.b b+ b b b
For $\nu\infty > 0$, we distinguish two cases: $v \equiv 0$ and v 6≡ 0. In the first case, we get $u \equiv 0$, a contradiction as before.
b

In the other case, we get $u > 0$ from (1.4) and $v > 0$ by repeating the same argument. Thus ν and u, v >b 0 satisfy the linear eigenvalue problem (b 2.1). ∞
However, we already discussed that (2.1) has precisely one eigenvalue $\nu_1 = \frac{\mu_1}{\sqrt{\theta_1\theta_2}}$ with componentwise positive eigenfunction $(\sqrt{\theta_1}\varphi_1,\sqrt{\theta_2}\varphi_1)$. Therefore, it must be that $\nu = \nu_1$ and ∞
$$(\widehat{u},\widehat{v}) = \frac{(\sqrt{\theta_1}\varphi_1,\sqrt{\theta_2}\varphi_1)}{\|(\sqrt{\theta_1}\varphi_1,\sqrt{\theta_2}\varphi_1)\|}.$$
This concludes the proof of Proposition 3.1.

Now we complete the proof of Lemma 2.1.The operator equation (3.2) satisfies the hypotheses of Proposition 2.2 with $F := L^+ + H$. Then there exist unbounded continua
$D_\pm \subset \widehat{\mathscr{S}} \overset{\text{def}}{=} \{(\lambda,(u,v)) \in \mathbb{R} \times E \mid (\lambda,(u,v))$ solution of (3.2)$\}$
containing (0,(0,0)). By the nonexistence result of Theorem 1.2 $D_+ \subset ([0,\lambda^*) \times E)$, and thus D_+ must be unbounded in the Banach space E-direction. Then $D \overset{\text{def}}{=} D_+ + D-$ is a continuum containing (0,(0,0)). By Proposition 3.1, ν_1 is the only bifurcation point from infinity for (3.2) and D_+ is unbounded in the E-direction, hence D_+ must bifurcate from infinity at ν_1. To conclude the proof of Lemma 2.1, it remains to verify that D satisfies the properties (i)–(iii).
It follows from assumption (A2) and (1.4)-(1.5) that $u,v > 0$ whenever $(\lambda,(u,v)) \in D$ and $\lambda > 0$, this implies part (i). For $\lambda = 0,(u,v) = (0,0)$ is the only solution of (2.2) and $(0,(0,0)) \in D$, hence part (ii) holds. Applying Proposition 2.2, we see that $D-$ must be unbounded in $\mathbb{R}\times E$. However, by part (ii) and the fact that ν_1 is the unique bifurcation point from infinity for (3.2), we see that $D-$ must be unbounded in the negative λ-direction, hence $(-\infty,\nu_1) \subset \text{Proj}_\lambda D$. This completes the proof of Lemma 2.1. 2 **Proof of Theorem 1.3** As discussed in Introduction, we prove the existence of a continuum C by taking the limit of sequence of continua corresponding to an asymptotically positively homogeneous system. In step 1, we discuss the approximation scheme, in step 2, we pass to the limit and in step 3, we give a prior bound.
Step 1. Approximation problems
Fix $n \in \mathbb{N}$ and define $f_n(s),g_n(s) : \mathbb{R} \to (0,\infty)$ by
defs $\leqslant n$ $f_n(s)$ =.
$$\begin{cases} f(s); & s \leqslant n \\ \frac{f(n)}{n}s; & n < s \end{cases}, \quad g_n(s) \overset{\text{def}}{=} \begin{cases} g(s); \\ \frac{g(n)}{n}s \end{cases}$$
; $n < s$
Then f_n and g_n are —— —— continuous functions on R.
For each $n \in \mathbb{N},(u,v) \in E \cap X$, we consider the following problem

, $$\begin{cases} u' + a(t)u = \lambda b(t) f_n(v), & t \in \mathbb{R} \\ v' + a(t)v = \lambda b(t) g_n(u), & t \in \mathbb{R}, \end{cases}$$

(3.4) which approaches (1.1) as $n \to +\infty$. We will use Lemma 2.1 to treat (3.4) and thus we rewrite (3.4) in the form

of system (2.2) as

, $$\begin{cases} u' + a(t)u = \lambda b(t)\frac{f(n)}{n} v^+ + \lambda b(t)\widetilde{f}_n(v), & t \in \mathbb{R} \\ v' + a(t)v = \lambda b(t)\frac{g(n)}{n} u^+ + \lambda b(t)\widetilde{g}_n(u), & t \in \mathbb{R}, \end{cases}$$

(3.5)

where $\widetilde{f}_n(x) \overset{\text{def}}{=} f_n(x) - \frac{f(n)}{n}x^+$ and $\widetilde{g}_n(x) \overset{\text{def}}{=} g_n(x) - \frac{g(n)}{n}x^+$. We note that $\widetilde{f}_n(x)$ and $\widetilde{g}_n(x)$ are bounded in R. Indeed, since $f_n(x)$ is nondecreasing and $f_n(x) = f(x) > 0$ for $x \leqslant n$, we get

$$|\widetilde{f}_n(x)| \leqslant \sup_{x\in\mathbb{R}} \left| f_n(x) - \frac{f(n)}{n}x^+ \right| \leqslant \max_{x\in[0,n]} \left| f_n(x) - \frac{f(n)}{n}x^+ \right| + f(0) = \text{const.} < +\infty,$$

where the constant is independent of x and depends on n. We can repeat the same argument for $\widetilde{g}_n$.

Since $f_n, g_n > 0$, it is easy to see that (3.5) satisfies the hypotheses of Lemma 2.1 with $\theta_1 = \frac{f(n)}{n}$ and $\theta_2 = \frac{g(n)}{n}$, $\widetilde{f} = \widetilde{f}_n, g = g_n$, e e and $\nu_1 = \nu_{1,n} \overset{\text{def}}{=} \frac{\mu_1 n}{\sqrt{f(n)g(n)}}$. Then by Lemma 2.1, $\nu_{1,n}$ is the unique bifurcation point from infinity for (3.5) and there exists a continuum C_n of positive solutions of (3.5) bifurcating from infinity at $\nu_{1,n}$ satisfying the properties (i)–(iii) of Lemma 2.1. In particular, $(0,(0,0)) \in C_n$, C_n is bounded above by the hyperplane $\lambda = \lambda_*$.

Step 2. Passing to the limit

Now we verify $\{C_n\}$ satisfying the conditions of Lemma 2.4. By the definition of the continuum, C_n is closed. Since all of C_n contain $(0,(0,0))$, we can choose $z_n \in C_n$ such that $z_n = (0,(0,0))$ for each $n = 1,2,\cdots$.

Naturally, $z_n \to z_* = (0,(0,0))$, the condition (i) of Lemma 2.4 is satisfied.

Obviously, because of the unboundedness of $\{C_n\}$, then $d_n = \sup\{|\mu| + \|(u,v)\| \mid (\mu,(u,v)) \in C_n\} = +\infty$, (ii) of Lemma 2.4 holds.

(iii) in Lemma 2.4 can be deduced directly from the Arzelà-Ascoli theorem and the definition of f_n, g_n. Therefore, the superior limit of $\{C_n\}$ contains a component $C \subset \Pi$ joining $(0,(0,0))$ with infinity, and it follows from $u,v > 0$ for $\lambda > 0$ whenever $(\lambda,(u,v)) \in C$, which establishes (a). Part (b) follows from $(0,(0,0)) \in C$ and $f(0), g(0) > 0$. (c) in Theorem 1.3 can be deduced directly from the Theorem 1.2.

Step 3. A prior bounds

Next we show for any closed and bounded interval $I \subset (0,\lambda_*)$, there exists $M > 0$, such that $\sup\{\|(x,y)\| \mid (\mu,(x,y)) \in C, \mu \in I\} \leqslant M$.

Suppose on the contrary that there exist $\{(\mu_n,(x_n,y_n))\} \subset C \cap (I \times K)$ with $\|(x_n,y_n)\| \to +\infty$.

This implies that for arbitrary $t \in \mathbb{R}$,

$$(x_n,y_n) = \mu_n Q(x_n,y_n) = \left(\mu_n \int_0^1 G(t,s) f(y_n(s-\tau(s)))ds, \mu_n \int_0^1 G(t,s) f(x_n(s-\tau(s)))ds\right),$$

where $Q : E \to E$ denotes the mapping $(x,y) \mapsto L^{-1}(f(y),g(x))$.

Let $(\widehat{x}_n,\widehat{y}_n) = \frac{(x_n,y_n)}{\|(x_n,y_n)\|}$, $\|(\widehat{x}_n,\widehat{y}_n)\| = 1$,

$(\; _n, \; _n) = \mu_n \frac{Q(x_n,y_n)}{\|(x_n,y_n)\|}$. $\widehat{x}$ $\widehat{y}$ then it follows

Choosing a subsequence of $\{(\mu_n,(x_n,y_n))\}$ and relabelling if necessary, it follows that there exists $(\mu_0,(x_0,y_0)) \in I \times E$ with $\|(x_0,y_0)\| = 1$ such thatb b

lim $(\mu_n y_n)) = (\mu_0,(x_0,y_0))$, $n\to+\infty$ $,(\widehat{x}_n,\widehat{}$

combines this with (H3) and the $G(t,s)$ positivity of, it follows that

$(x_0,y_0) = \mu_0(+\infty,+\infty)$,

this contradicts $\|(\widehat{x}_n,\widehat{y}_n)\| = 1$, therefore, $\sup\{\|(x,y)\|$

$| (\mu,(x,y)) \in C, \mu \in I\} \leqslant M$.

Finally, we prove C must bifurcate from infinity at $\lambda \to 0^+$. Now let $(\lambda_n,(u_n,v_n)) \in C$ with $\|(\mu_n,(u_n,v_n))\| \to +\infty$ as $n \to +\infty$ and $\lambda_n > 0$ for all $n \in \mathbb{N}$. Suppose to the contrary that $\lambda_n \to \lambda' > 0$ as $n \to +\infty$, then there exists a closed and bounded interval I such that $\lambda' \in I$. By the above proof, $\|(u_n,v_n)\| \leqslant M < +\infty$ for all λ', a contradiction to $\|(u_n,v_n)\| \to +\infty$ as $n \to +\infty$, which establishes part (d) and completes the proof of Theorem 1.3.

3. Conclusion

In this paper, the global structure of positive periodic solutions for a class of superlinear first-order periodic differential systems is studied by using the global bifurcation theory. The innovation of this paper is that we use a new technique to provide the global behavior of the set of positive periodic solutions without the additional conditions mentioned in previous papers.

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