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SYMMETRY'S HIDDEN CONNECTIONS: GROUP THEORY'S INFLUENCE ACROSS CHEMISTRY, MATHEMATICS, AND PHYSICS

Edward G
NOVA, Department of Mathematics, 8333Little River Turnpike, Annandale, VA 22003, USA
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Abstract: Symmetry and group theory play a pivotal role in various scientific disciplines, facilitating an understanding of molecular properties, mathematical topologies, and the fundamental symmetries of physical laws. This report explores the interplay between symmetry, group theory, and energy laws in diverse scientific contexts, ranging from chemistry and mathematics to physics. By harnessing the power of group theory, we unveil profound insights into the nature of molecules, topological spaces, and the fundamental laws governing the universe. In chemistry, the connection between molecular symmetry and physical attributes is established. The symmetry of molecules offers a powerful tool to predict energy levels, orbital symmetries, transition possibilities, and bond orders without resorting to intricate calculations. Meanwhile, the realm of mathematics introduces the concept of the fundamental group, where topological properties find correspondence in algebraic structures. This mathematical bridge enhances our comprehension of proximity, continuity, and their reflection in group properties. Turning to the realm of physics, symmetry groups emerge as a cornerstone for elucidating the symmetries underpinning the laws of the universe. Noether's theorem establishes a profound link between continuous symmetries and conservation laws in physical systems. Exemplifying this, the Standard Model, gauge theory, and groups like the Lorentz and Poincaré groups play pivotal roles in describing fundamental interactions. Notably, topology, group theory, and energy conservation intertwine harmoniously across scientific domains.
Central to our argument is the utilization of homology group chains as energy carriers within algebraic topology. These chains illuminate the underlying topological structures within corresponding spaces. Just as a polycrystal consists of crystallites with distinct orientations, a topological space can be envisaged as a mosaic of interconnected simplices. The arrangement of boundaries between these simplices orchestrates phenomena akin to light scattering, accentuating the role of connectivity in topological spaces. We contend that abrupt changes in this connectivity, analogous to fragmentation, can impede global topological flows, leading to isolated neighborhoods and disrupted continuity.
In essence, this report underscores the profound influence of symmetry, group theory, and topological insights on the understanding of molecular properties, mathematical structures, and the fundamental laws of physics. By delving into these interdisciplinary connections, we enrich our grasp of the underlying fabric of the cosmos.

Keywords: symmetry, group theory, algebraic topology, fundamental group, conservation laws.

I. INTRODUCTION

In Chemistry, groups and group theory show how the symmetry of a molecule is related to its physical properties and provides a quick simple method to determine the relevant physical information of the molecule. The symmetry of a molecule provides you with the information of what energy levels the orbitals will be, what the orbitals symmetries are, what transitions can occur between energy levels, even bond order to name a few can be found, all without rigorous calculations. In mathematics, algebraic topology, we introduce the fundamental group, which by means of this connection, topological properties such as proximity and continuity translate into properties of groups.In physics, groups are important because they describe the symmetries which the laws of physics seem to obey. According to Noether's theorem, every continuous symmetry of a physical system corresponds to a conservation law of the system. Examples of the use of groups in physics include the Standard Model, gauge theory, the Lorentz group, and the Poincaré group. And more generally, the interplay between topology, group theory and energy laws is evident in several areas of science. In this report, we would like to claim that homology group chains are used in algebraic topology as energy carriers for topological underlying structures in corresponding topological spaces, and the details of the chains are the necessary and determining pieces of information for the topological configurations of the underlying spaces.

In a certain way, we can think of a topological space (or a simplicial complex, as we will see later) as a polycrystal, an object composed of randomly oriented crystalline regions, called crystallites (topological simplices). The arrangement of the boundaries between individual crystallites in a polycrystal causes them to scatter a beam of light instead of reflecting or refracting it uniformly, so that even colorless polycrystals are opaque (in the case of topological space, the concept of light corresponds to the connectivity of the space as thedegree to which a neighborhood of a point facilitates or impedes “topological” flows (e.g., the “movement” of geometry or topological properties among neighborhoods). An abrupt change in the connectivity of the landscape, for example, as might be caused by fragmentation, may interfere with widespread global topologicalpropertiesand may lead to fragmented, small, isolated neighborhoods. Similar discussions can follow with respect to connectedness as a measure of continuityof the topological type.

II. ISOMORPHISMS

The concept of an isomorphism is fundamental and foundational in mathematics, as it appears in several areas of mathematics. The origin/etymology of the word is the Greek iso, meaning "equal," and morphosis, meaning “to form” or “to shape.” A structure preserving map between two algebraic structures is called a homomorphism. We understand an isomorphism to be a one to one and onto (bijective) map (morphism) that preserves sets and relations among elements (homomorphism). This is important since we need structure-preserving mapping between two sets with structures of the same type that can be reversed by an inverse mapping. We say that “A is isomorphic to B” is written A=B. Moreover, an isomorphism from a set of elements onto itself is called an automorphism. A canonical isomorphism is an isomorphism if this is the only one isomorphism between the two structures, or if the isomorphism is much more natural (in some sense) than other isomorphisms. There is a variety of theorems and applications of isomorphisms and automorphisms, such as the Ax-Kochen Isomorphism Theorem, the concept of a Graph Isomorphism, the concept of a Homeomorphism, the concept of Isomorphic Groups, etc. Here we will discuss a few of these ideas. One important aspect of an isomorphism is that two isomorphic objects have the same properties with respect to a certain structure and therefore isomorphic structures cannot be distinguished from the point of view of structure only, and may be identified, and we say that two objects are the same up to an isomorphism. One way to understand the operation of an isomorphism is to think as space to be a mirror and an isomorphism is the operation of mirroring.

There is a variety of names assigned to isomorphisms depending on the type of structure under consideration, be it geometric, topological, differential, graphical, etc. For example: An isometry is an isomorphism of metric spaces. A homeomorphism is an isomorphism of topological spaces. A diffeomorphism is an isomorphism of spaces equipped with a differential structure, typically differentiable manifolds. A permutation is an automorphism of a set. In geometry, isomorphisms and automorphisms are often called transformations, for example rigid transformations, affine transformations, projective transformations.

Category theory is a way to formalize mathematical structure and its concepts in terms of labeled directed graphs (categories), whose nodes are called objects (spaces, groups, etc.), and whose labelled directed edges are called arrows (or morphisms, isomorphisms, etc.) [1] A category has two basic properties: the ability to compose the arrows associatively, and the existence of an identity arrow for each object. The language of category theory has been used to formalize concepts of other high-level abstractions such as sets, rings, and groups. Categories now appear in many branches of mathematics, some areas of theoretical computer science where they can correspond to types or to database schemas, and mathematical physics where they can be used to describe vector spaces [2], as well as in biology in the "metabolism-repair" model of autonomous living organisms by Robert Rosen [3]. Category theory may also be used as an axiomatic foundation for mathematics, as an alternative to set theory and other proposed foundations.

The definition of an isomorphism is as follows. If one object consists of a set X with a binary relation R and the other object consists of a set Y with a binary relation S then an isomorphism from X to Y is a bijective function $f\colon X \to Y$ such that:

$$S(f(u), f(v)) \Leftrightarrow R(u, v), \forall u, v \in X$$

Therefore, S is reflexive, irreflexive, symmetric, antisymmetric, asymmetric, transitive, total, trichotomous, a partial order, total order, well-order, strict weak order, total preorder (weak order), an equivalence relation, or a relation with any other special properties, if and only if R is.

In algebra, isomorphisms are defined for all algebraic structures, for example: Linear isomorphisms between vector spaces, specified by invertible matrices. Group isomorphisms between groups that help the classification of isomorphism classes of finite groups is an open problem. Ring isomorphism between rings. Field isomorphisms are the same as ring isomorphism between fields (Galois theory). Just as the automorphisms of an algebraic structure form a group, the isomorphisms between two algebras sharing a common structure form a heap. Letting a particular isomorphism identify the two structures turns this heap into a group. In mathematical analysis, the Laplace transform is an isomorphism mapping hard differential equations into easier algebraic equations. In graph theory, an isomorphism between two graphs G and H is a bijective map f from the vertices of G to the vertices of H that preserves the "edge structure" in the sense that there is an edge from vertex u to vertex v in G if and only if there is an edge from $f(u)$ to $f(v)$ in H. In mathematical analysis, an isomorphism between two Hilbert spaces is a bijection preserving addition, scalar multiplication, and inner product In early theories of logical atomism, the formal relationship between facts and true propositions was theorized by Bertrand Russell and Ludwig Wittgenstein to be isomorphic. A bijective continuous function whose inverse is also continuous is an isomorphism between topological spaces, called a homeomorphism. Bijective morphisms are not necessarily isomorphisms (such as the category of topological spaces).

III. HOMOLOGY GROUPS

In an effort to classify spaces topologically, one principle used is the idea of finding a region without boundaries, which is not itself a boundary of some region. This principle is mathematically elaborated into the theory of homology groups. The mathematical structures underlying homology groups are finitely generated Abelian groups. We start by representing each part of a figure by some standard object. We take triangles and their analogues in other dimensions, called simplexes, as the standard objects. Simplexes are building blocks of a polyhedron. A 0simplex is a point, or a vertex, a 1-simplex is is a line segment or an edge. A 2-simplex is a triangle with its interior and a 3 simplex is a solid tetrahedron, and so on and so forth. By this standardization, it becomes possible to assign each figure Abelian group structures [4]. This originates from the definition of a simplicial complex. If we let K be a set of finite number of simplexes in $\mathbb{R}_m$, then K is a simplicial complex if certain conditions are met: (a) an arbitrary face of a simplex of K belongs K , that is, $\sigma \in K$and if $\acute{\sigma} \le \sigma$, then $\acute{\sigma} \in K$, and (b) if $\acute{\sigma}, \sigma \in K$ are simplexes of K, the intersection $\acute{\sigma} \cap \sigma$ is either empty or common face of $\acute{\sigma}, \sigma$, $\acute{\sigma} \cap \sigma \le \sigma$ and $\acute{\sigma} \cap \sigma \le \acute{\sigma}$. After introducing an appropriate concept of orientation of a simplex, we can define the r-chain group $C_r(K)$ of a simplicial complex K as a free Abelian group generated by r-simplexes of K. If $r > \dim K$, $C_r(K)$ is defined to be 0. An element of $C_r(K)$ is called an rchain. A boundary operator ∂_r is defined as follows:

$$\partial_r : C_r(K) \longrightarrow C_{r-1}(K)$$

$$\partial_r \sigma_r \equiv \sum_{i=0}^{r} (-1)^r (p_0 p_1 \ldots \hat{p}_i \ldots p_r)$$

Where $\sigma_r = (p_0 p_1 \ldots p_r)$

Then we can define Kernel of this operator

$$\mathrm{Ker}\, \partial_r = Z_r(K) = \{c \in C_r(K) \mid \partial_r c = 0\}$$

and image of the operator

$$\mathrm{Im}\, \partial_{r+1} = B_r(K)$$
$$= \{c \in C_r(K) \mid c = \partial_r d \text{ for some } d \in C_r(K)\}$$

and the composite map

$$\partial_r \partial_{r+1} = 0.$$

One can observe that $B_r(K) \subset Z_r(K) \subset C_r(K)$. These three groups associated with a simplicial complex K are actually related to topological properties of K or to the topological space whose triangulation is K. The r-th homology group

$$H_r(K) \equiv Z_r(K) / B_r(K) = \{[z] \mid z \in Z_r(K)\},$$

where each equivalence class [z] is called a homology class. These homology groups are actually topological invariants. One can show that, if X is homeomorphic to Y and let (K, f) and (L, g) be triangulations of X and Y respectively, then we have

$H_r(K) \cong H_r(L)$. Because of this property, we can see that if we have two triangulations of the same space X, we will have the same homology groups. Therefore we can define the homology groups of a topological space X which is not necessarily a

polyhedron but which is triangulable

$$H_r(X) \equiv H_r(K) \quad r = 0, 1, 2, \ldots$$

for an arbitrary triangulation (K, f).

Regarding the structure of homology groups, it can be shown that the most general form of $H_r(K)$ which is not necessarily a free Abelian group as $Z_r(K)$ and

$B_r(K)$, is the following:

$$H_r(K) \cong \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \oplus \mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_p}$$

The number of generators of $H_r(K)$ counts the numbers of (r+1) dimensional holes in $|K|$and the first factors of form a free Abelian group of rank f and the next p factors are called the torsion subgroup of

$H_r(K)$, which, in a sense, detects the “twisting” in the polyhedron $|K|$, the subset of $\mathbb{R}_m$made out of the union of all simplices of K, for a homology with integer coefficients.

In dynamical systems theory in physics, Poincaré was one of the first to consider the interplay between the invariant manifold of a dynamical system and its topological invariants. Morse theory relates the dynamics of a gradient flow on a manifold to, for example, its homology. Floer homology extended this to infinite-dimensional manifolds. The KAM theorem established that periodic orbits can follow complex trajectories; in particular, they may form braids that can be investigated using Floer homology. [5]

IV. CONSERVATION OF ENERGY

In physics and chemistry, the law of conservation of energy is a statement that the total energy of an isolated system remains constant; it is said to be conserved over time. This law, first proposed and tested by Émilie du Châtelet, means that energy can neither be created nor destroyed; rather, it can only be transformed or transferred from one form to another. Classically, conservation of energy was distinct from conservation of mass; however, special relativity showed that mass is related to energy and vice versa by E = mc2, and science now takes the view that mass–energy as a whole is conserved.

Conservation of energy can be rigorously proven by Noether's theorem as a consequence of continuous time translation symmetry; that is, from the fact that the laws of physics do not change over time. More precisely: Let there be a set of differentiable fields φ defined over all space and time; the action is now an integral over space and time

$$S = \int \mathscr{L}(\varphi, \partial_\mu \varphi, x^\mu) d^4x$$

A continuous transformation of the fields can be written infinitesimally as

$$\varphi \rightarrow \varphi + \varepsilon_r \Psi_r$$

$$\mathscr{L} \rightarrow \mathscr{L} + \varepsilon_r \partial_\mu \Lambda_r^\mu$$

With conserved densities

$$j_r^\nu = \Lambda_r^\nu - \frac{\partial \mathscr{L}}{\partial \varphi_{,\nu}} \cdot \Psi_r$$

With conservation law

$$\partial_\nu j^\nu = 0$$

For systems which do not have time translation symmetry, it may not be possible to define conservation of energy. Examples include curved spacetimes in general relativity [6] or time crystals in condensed matter physics [7].

Historically, the concept of conservation of energy can be traced back to ancient philosophers such as Thales of Miletus (c. 550 BCE) (conservation of water as some underlying substance of which everything is made), Empedocles (490–430 BCE) who wrote that in his universal system, composed of four roots (earth, air, water, fire), "nothing comes to be or perishes"[8], Epicurus (c. 350 BCE) who believed that everything in the universe to be composed of indivisible units of matter and stated that "the sum total of things was always such as it is now, and such it will ever remain."[9]

Other contributions come from Simon Stevinus in 1605(perpetual motion in statics is impossible),

Galileo in 1639 who published his analysis on the "interrupted pendulum" conservatively converting potential energy to kinetic energy and back again. Christiaan Huygens in 1669 published his laws of collision, where among the quantities he listed as being invariant before and after the collision of bodies were both the sum of their linear momenta as well as the sum of their kinetic energies. Gottfried Leibniz, during 1676–1689 attempted a mathematical formulation of the kind of energy which is connected with motion (kinetic energy).

Émilie du Châtelet (1706–1749) proposed and tested the hypothesis of the conservation of total energy, as distinct from momentum. Inspired by the theories of Gottfried Leibniz. Earlier workers, including Newton and Voltaire, had all believed that "energy" (so far as they understood the concept at all) was not distinct from momentum and therefore proportional to velocity. Du Châtelet proposed that energy must always have the same dimensions in any form, which is necessary to be able to relate it in different forms (kinetic, potential, heat…)[10].

In a paper Über die Natur der Wärme(German "On the Nature of Heat/Warmth"), published in the ZeitschriftfürPhysik in 1837, Karl Friedrich Mohr gave one of the earliest general statements of the doctrine of the conservation of energy: "besides the 54 known chemical elements there is in the physical world one agent only, and this is called Kraft [energy or work]. It may appear, according to circumstances, as motion, chemical affinity, cohesion, electricity, light and magnetism; and from any one of these forms it can be transformed into any of the others."

The conservation of energy is, from a mathematical point of view is a consequence of Noether's theorem, developed by Emmy Noether in 1915 and first published in 1918. The theorem states every continuous symmetry of a physical theory has an associated conserved quantity; if the theory's symmetry is time invariance then the conserved quantity is called "energy". Energy conservation is implied by the empirical fact that the laws of physics do not change with time itself. In other words, if the physical system is invariant under the continuous symmetry of time translation then its energy (which is canonical conjugate quantity to time) is conserved. Conservation of energy for finite systems is valid in such physical theories as special relativity and quantum theory (including QED) in the flat spacetime. With the discovery of special relativity by Henri Poincaré and Albert Einstein, energy was proposed to be one component of an energy-momentum 4-vector. Each of the four components (one of energy and three of momentum) of this vector is separately conserved across time, in any closed system, as seen from any given inertial reference frame. Also conserved is the vector length (Minkowski norm), which is the rest mass for single particles, and the invariant mass for systems of particles.The relativistic energy of a single massive particle contains a term related to its rest mass in addition to its kinetic energy of motion. In the limit of zero kinetic energy (or equivalently in the rest frame) of a massive particle, the total energy of particle or object is related to its rest mass or its invariant mass via the famous equation {\displaystyle E=mc^{2}}E=mc^{2}.The rule of conservation of energy over time in special relativity continues to hold, so long as the reference frame of the observer is unchanged. This applies to the total energy of systems, although different observers disagree as to the energy value. In general relativity, energymomentum is typically expressed with the aid of a stress–energy–momentum pseudotensor. If the metric under consideration is static or asymptotically flat, then energy conservation holds, however some metrics such as the Friedmann–Lemaître–Robertson– Walker metric do not satisfy these constraints and energy conservation is not well defined.[24] The theory of general relativity leaves open the question of whether there is a conservation of energy for the entire universe.

V. CONCLUSION

In physics, we understand a conservation law as a statement that a particular measurable property/quantity of an isolated physical system does not change as the system evolves over time and that there is a mathematically expressed "transport" mechanism of that quantity. So, the concept of time as a parameter of evolution or variation of the state of the system is foundational in the concept of conservation. We see examples of conservation laws in energy, in linear momentum, in angular momentum, in electric charge, etc. Approximate conservation laws include laws which apply to such quantities as mass, parity, lepton number, baryon number, strangeness, hypercharge, etc. However, according to Noether's theorem, there is a one-to-one correspondence between each conservation law and a differentiable symmetry of nature. The conservation of energy follows from the time-invariance of physical systems, and the conservation of angular momentum arises from the fact that physical systems behave the same regardless of how they are oriented in space. Our position is that a conservation law can be related to the topology of a space. We have an examplesin physics and biology of, magnetic topology (the structure of linkage of magnetic flux) of an ideal plasma and a discussion of its conservation [11], conservation and functional significance of gene topology in the genome [12], conservation of topology, But Not conformation, of the Proteolipid Proteins of the Myelin Sheath [13], etc. , It has also been shown that, like Noether current fields, there are topological current fields (derivatives of solitary waves derived from a degenerate vacuum) and the corresponding conservation laws associated with these current-fields relate to a common mathematical structure which is the Rhamcohomology [14]. Therefore, we would like to propose that the topological structure of a topological space is the "state" of the system, the homology chain is the "energy" of the system, the "transport" is realizable through not time but rather a homeomorphism, and the "transport mechanism" is the homology chain isomorphism. In other words, the "energy of the topology" is mirrored, analyzed and modeled through the homology chain. The homology is not "equal" from one state of the system to the other, but rather "isomorphic". We understand the structure of the group chain is primary source of energy for a topological system and it is responsible for the distribution of energy throughout the topological space by classifying and relating the simplicial complex. Similar ideas have been discussed before, as in the case of distribution grids that constitute complex networks of lines oftentimes reconfigured to minimize losses, balance loads, to alleviate faults, or for maintenance purposes [15], where topology monitoring becomes a critical task for optimal grid scheduling.

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